The quantum brachistochrone problem for non-Hermitian Hamiltonians

The quantum brachistochrone problem for non-Hermitian Hamiltonians

Paulo E G Assis and Andreas Fring

Centre for Mathematical Science, City University, Northampton Square, London EC1V 0HB, UK

E-mail: Paulo.Goncalves-De-Assis.1@city.ac.uk and A.Fring@city.ac.uk

Abstract

Recently Bender, Brody, Jones and Meister found that in the quantum brachistochrone problem the passage time needed for the evolution of certain initial states into specified final states can be made arbitrarily small, when the time-evolution operator is taken to be non-Hermitian but $\mathcal{PT}$-symmetric. Here we demonstrate that such phenomena can also be obtained for non-Hermitian Hamiltonians for which $\mathcal{PT}$-symmetry is completely broken, i.e. dissipative systems. We observe that the effect of a tunable passage time can be achieved by projecting between orthogonal eigenstates by means of a time-evolution operator associated with a non-Hermitian Hamiltonian. It is not essential that this Hamiltonian is $\mathcal{PT}$-symmetric.

PACS numbers: 03.65.Xp, 03.65.Ca, 03.65.−w, 11.30.Er

1. Introduction

To find the brachistochrone is one of the oldest problems in classical mechanics tracing back to Newton and Leibniz. It consists of finding the trajectory between two locations of a particle, subject to a gravitational field, for which the transition time becomes minimal. This problem can be generalized to a relativistic [1] and to a quantum mechanical setting [2–6]. In the latter case one seeks the minimal time $t =: \tau$, referred to as passage time, such that

$$|\psi_f\rangle = e^{-i\tau H}|\psi_i\rangle, \tag{1}$$

for given initial and final states $|\psi_i\rangle$ and $|\psi_f\rangle$, respectively. Equality can be achieved by possibly tuning some parameters in the Hamiltonian H.

Bender, Brody, Jones and Meister [6] extended this treatment by also allowing non-Hermitian Hamiltonians in (1). The surprising result found in [6] was that when involving non-Hermitian, but $\mathcal{PT}$-symmetric Hamiltonians, in the evolution operator, the passage time can be made arbitrarily small by varying a parameter in H while keeping the transition frequency between two states constant. At present this phenomenon is an observation and no explanation has been provided as to where this effect might originate from.

One might suspect that one could make the $\mathcal{PT}$-symmetry responsible for the observation and seek similar arguments as those which allow one to explain the reality of the spectrum of a $\mathcal{PT}$-symmetric non-Hermitian Hamiltonian. See for instance [7–10] for recent results and reviews. The main purpose of this paper is to investigate whether $\mathcal{PT}$-symmetry can be utilized as well to explain the observed phenomenon of a tunable passage time. In fact, we find that the same conclusion can be drawn when considering non-Hermitian Hamiltonians with complex eigenvalues describing dissipative systems, i.e. those for which $\mathcal{PT}$-symmetry is definitely broken. This means the possibility of arbitrarily small passage times results from the non-Hermitian nature of the Hamiltonian involved in the time-evolution operator and not its $\mathcal{PT}$-invariance.

Our manuscript is organized as follows: in section 2 we derive passage times for $\mathcal{PT}$-symmetric Hamiltonians for various different types of initial and final states. In section 3 we perform a similar analysis for non-Hermitian Hamiltonians with complex eigenvalues deriving similar phenomena to those section 2. We state our conclusions in section 4.

2. Pseudo-Hermitian Hamiltonians

We start by considering $\mathcal{PT}$-symmetric or rather pseudo-Hermitian Hamiltonians. We recall [11–15] that a non-Hermitian Hamiltonian H is said to be a pseudo-Hermitian operator, if there exists a Hermitian operator η, with regard to the standard inner product, such that

$$H^\dagger = \eta^2 H \eta^{-2} \quad \Leftrightarrow \quad h = \eta H \eta^{-1} = h^\dagger. \tag{2}$$

The virtue of such a conjugate pair h and H is that they possess an identical eigenvalue spectrum, because the Hamiltonians lie in the same similarity class. The reality of the spectrum is guaranteed, since one of the Hamiltonians involved, i.e. h, is Hermitian. The solutions of the corresponding time-independent Schrödinger equations $H\Phi = \varepsilon\Phi$ and $h\phi = \varepsilon\phi$ are then simply related as

$$\Phi = \eta^{-1}\phi. \tag{3}$$

Let us first discuss the quantum brachistochrone problem for these types of systems.

2.1. $\phi \to \phi$ via $u \equiv \Phi \to \Phi$ via U

Taking the initial state $|\psi_i\rangle$, the final state $|\psi_f\rangle$ to be orthonormal states of a Hermitian Hamiltonian system and the time-evolution operator in (1) to be Hermitian as well is the simplest situation to investigate. Equivalent would be to investigate the non-Hermitian system obtained by the similarity transformations (2) and (3). Here we want to solve the quantum brachistochrone problem in a slightly less stringent form as studied in [2–6]. Instead of solving (1), we just consider the physical relevant matrix element and seek the minimal time $t =: \tau$, such that a given transition probability is reached. This means for normalized initial and final states we solve the equation

$$|\langle \phi_f | u(\tau, 0)\phi_i \rangle| = |\langle \Phi_f | U(\tau, 0)\Phi_i \rangle_\eta| = \beta, \tag{4}$$

for τ with given constant $0 \leqslant \beta \leqslant 1$. Here $u(t, t')$ and $U(t, t') = \eta^{-1}u(t, t')\eta$ are the time-evolution operators, which evolve a wavefunction from time t' to t associated with the Hermitian and non-Hermitian system, respectively. In order to make the inner product involving the Φs meaningful, we have to change the metric in the second expression in (4) as was argued in [11, 13, 16] or equivalently use a biorthonormal basis, see e.g. [17, 18]. Here we use the convention $\langle \psi_f | \mathcal{O}\psi_i \rangle_\eta := \langle \psi_f | \eta^2 \mathcal{O}\psi_i \rangle$ for the η-inner product with $\mathcal{O}$ being some operator. Clearly, for given time-evolution operators and final and initial states a real solution

for τ for all values of β does not always exist. Natural choices are for instance $\beta = 1$ or the maximum transition amplitude.

Starting with time independent Hamiltonians, the problem is solved in a straightforward manner as we simply have $u(\tau, 0) = e^{-i\tau h}$ and $U(\tau, 0) = e^{-i\tau H}$. When ϕ_i and ϕ_f are orthogonal states we can find a solution in complete generality. Taking $|\phi_+\rangle$ and $|\phi_-\rangle$ to be two normalized eigenstates of h, the two states

$$|\phi_{f/i}\rangle = \frac{1}{\sqrt{2}}(|\phi_-\rangle \pm i|\phi_+\rangle). \tag{5}$$

are orthonormal with regard to the standard inner product. It is then straightforward to compute the matrix element occurring in (1)

$$\begin{aligned}\langle\phi_f|e^{-ith}\phi_i\rangle &= \frac{1}{\sqrt{2}}\langle\phi_f|(e^{-it\varepsilon_-}\phi_-) - i|e^{-it\varepsilon_+}\phi_+\rangle) \\ &= \frac{1}{2}e^{-i\varepsilon_- t}(1 - e^{-i\omega t}), \end{aligned} \tag{6}$$

where the transition frequency between the two states is denoted as $\omega = \varepsilon_+ - \varepsilon_-$. This means the quantum brachistochrone problem in the version (1) is solved for the passage time

$$\tau = \frac{2}{\omega}\arcsin\beta. \tag{7}$$

For $\beta = 1$ we recover $\tau = \pi/\omega$, which in slightly different forms, is well known and holds for any Hermitian or equivalent non-Hermitian system, as specified in (2). As pointed out first in [6] more spectacular results can be obtained when involving non-Hermitian Hamiltonians in the evolution operator while keeping the eigenstates to be associated to a Hermitian system.

2.2. $\phi \to \phi$ via $U \equiv \Phi \to \Phi$ via u

In [6] a Gedanken experiment was proposed in which a particle passes through a region, which causes its governing Hamiltonian to change from a Hermitian to a non-Hermitian one. This scenario implies that the Hamiltonian becomes explicitly time-dependent. The situation considered in [6] was for the initial and final states to be orthogonal states in a Hermitian system, whereas the time evolution was associated with a non-Hermitian Hamiltonian. In general, we can write this temporary change of the Hamiltonian to a non-Hermitian Hamiltonian in the form

$$H(t) = h + gh_1(t), \tag{8}$$

where $h = h^\dagger$ and $h_1 \neq h_1^\dagger$. This means we consider an analogue to a conventional time dependent scenario, however, with the difference that the perturbation is now non-Hermitian. A standard example of $H(t)$ in (8) with h_1 being Hermitian is for instance the Stark–LoSurdo Hamiltonian describing an atom in an external electric field, with h representing the unperturbed atomic system and $h_1(t)$ the external electric field. In [7, 19] the alternative scenario was considered in which also the unperturbed system was taken to be non-Hermitian. The treatment in [6] corresponds to the special case of (8) in which the time dependence is of the form of a stepfunction. This means to describe that setting one has to take $H(t) = H = h + gh_1$ for $0 \leqslant t \leqslant \tau$ and $H(t) = h$ for $t > \tau$, with τ being the passage time rather than the pulse length as in the aforementioned example.

Then, depending on the choice of the initial and final states, the time-evolution operator and the inner product, the quantum brachistochrone problem can be formulated in various different ways from (4). For instance, when projecting between orthogonal Hermitian states via a non-Hermitian time-evolution operator one may consider

$$\frac{|\langle\phi_f|U(\tau, 0)\phi_i\rangle_\eta|}{\|\phi_f\|_\eta\|\phi_i\|_\eta} = \frac{|\langle\eta\phi_f|u(\tau, 0)\eta\phi_i\rangle|}{|\eta\phi_f||\eta\phi_i|} = \beta, \tag{9}$$

where we used the η-norm defined as $\|\phi\|_\eta := |\eta\phi| = \sqrt{\langle\eta\phi|\eta\phi\rangle}$. This switching between the eigenstates is in fact the key point of the entire analysis. We may view equation (9) in two equivalent ways. On one hand on the right-hand side we just project with a standard Hermitian time-evolution operator, between two somewhat unusual, but perfectly viable initial and final states $\eta\phi_i$ and $\eta\phi_f$, respectively. This is a picture entirely in the standard quantum mechanical description with the difference that the initial and final states are not taken to be orthogonal. On the other hand we may use the equality and view this expression as a projection of some orthogonal initial and final states by means of a non-Hermitian time-evolution operator. We stress the fact that the initial and final states do not propagate in time and that the metric is not changed in this procedure[1].

Slightly different and less natural is the possibility corresponding to (1) when projecting with the standard inner product onto a final state. When written as an expectation value this amounts to

$$\frac{|\langle\phi_f|U(\tau,0)\phi_i\rangle|}{\|\phi_f\|_{\eta^{-1}}\|\phi_i\|_\eta} = \frac{|\langle\eta\phi_f|u(\tau,0)\eta\phi_i\rangle_{\eta^{-1}}|}{|\eta^{-1}\phi_f||\eta\phi_i|} = \beta. \tag{10}$$

Alternatively, one may also envisage a situation when one projects from orthogonal Hermitian states onto eigenstates of a non-Hermitian Hamiltonian via a non-Hermitian time-evolution operator, or vice versa. Then one should solve

$$\frac{|\langle\Phi_f|U(\tau,0)\phi_i\rangle_\eta|}{\|\Phi_f\|_\eta\|\phi_i\|_\eta} = \frac{|\langle\phi_f|u(\tau,0)\eta\phi_i\rangle|}{|\phi_f||\eta\phi_i|} = \beta, \tag{11}$$

or

$$\frac{|\langle\phi_f|U(\tau,0)\Phi_i\rangle_\eta|}{\|\phi_f\|_\eta\|\Phi_i\|_\eta} = \frac{|\langle\eta\phi_f|u(\tau,0)\phi_i\rangle|}{|\eta\phi_f||\phi_i|} = \beta. \tag{12}$$

As in a conventional time-dependent scenario one is rarely able to compute the time-evolution operator exactly. However, assuming the non-Hermitian term in (8) to be small when compared with h, we may apply standard perturbation theory by iterating the DuHamel formula [20–22]

$$U_H(t,t') = U_h(t,t') - \mathrm{i}g \int_{t'}^{t} \mathrm{d}s\, U_H(t,s)h_1(s)U_h(s,t').$$

Taking $U_h(t,t') = \exp(-\mathrm{i}h(t-t'))$ we obtain to first order in g

$$U_H(t,0) = \mathrm{e}^{-\mathrm{i}ht} - \mathrm{i}g \int_0^t \mathrm{d}s\, \mathrm{e}^{-\mathrm{i}h(t-s)}h_1(s)\,\mathrm{e}^{-\mathrm{i}hs}. \tag{13}$$

We may then compute for instance perturbatively the matrix element

$$\langle\phi_f|U_H(t,0)\phi_i\rangle = (1 - \mathrm{e}^{-\mathrm{i}\omega t})\left[\tfrac{1}{2} + \mathrm{i}g\langle\phi_+|\mathrm{Im}\,h_1\phi_-\rangle\right].$$

When h_1 is Hermitian we naturally recover the result in (6). One may now proceed perturbatively using the above expression for $U_H(t,0)$. However, it is clear from the previous discussion that essentially all aspects of the problem, which we wish to consider here, may be illustrated by selecting a two-level system from the larger, possibly even infinite, spectrum. Thus without loss of generality one may consider a 2×2 matrix Hamiltonian.

In order to set the scene for the next section let us briefly recall with some minor variation the analysis of [6].

---

[1] In the *preprint* quant-ph/0701223v2, *Is PT-symmetric quantum mechanics just quantum mechanics in a non-orthogonal basis?* by D Martin the result of Bender *et al* [6] has been challenged. However, the author has just missed this point and only argued on the two equivalent formulations of equation (4), which evidently always yields the same passage time (7). Some time after the publication of our preprint a further manuscript appeared, which also challenges the results in [6], namely *preprint* quanth-ph/0706.3844 *Quantum Brachistochrone Problem and the Geometry of the State Space in Pseudo-Hermitian Quantum Mechanics* by A Mostafazadeh. The author uses interesting geometrical arguments, but makes the same mistake as D. Martin by arguing in the setting of equation (4) and not addressing the crucial equation (9).

2.3. A 2 × 2 *matrix Hamiltonian*

A pair of 2 × 2 matrix Hamiltonians related by a similarity transformation as in (2) is

$$H = \begin{pmatrix} r\,\mathrm{e}^{\mathrm{i}\theta} & s \\ s & r\,\mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix}, \qquad h = \begin{pmatrix} r\cos\theta & -\frac{\omega}{2} \\ -\frac{\omega}{2} & r\cos\theta \end{pmatrix}, \tag{14}$$

with $\omega = 2\sqrt{s^2 - r^2\sin^2\theta}$ and $r, s, \theta \in \mathbb{R}$. For the eigenvalues $\varepsilon_\pm = r\cos\theta \pm \omega/2$ to be real, one requires $s^2 \geqslant r^2\sin^2\theta$, such that it is meaningful to introduce the new parameterization $\sin\alpha = r/s\sin\theta$ with $\alpha \in \mathbb{R}$. This parameter range guarantees therefore unbroken $\mathcal{PT}$-symmetry. The Hamiltonians h and H are related by the similarity transformation in (2), involving the Hermitian operator

$$\eta = \frac{1}{\sqrt{\cos\alpha}} \begin{pmatrix} \sin\alpha/2 & -\mathrm{i}\cos\alpha/2 \\ \mathrm{i}\cos\alpha/2 & \sin\alpha/2 \end{pmatrix}. \tag{15}$$

The normalized eigenstates of H and h are

$$|\Phi_\pm\rangle_\alpha = \frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{4}(1\mp1)}}{\sqrt{2\cos\alpha}} \begin{pmatrix} -\mathrm{e}^{\pm\frac{\mathrm{i}\alpha}{2}} \\ \mp\mathrm{e}^{\mp\frac{\mathrm{i}\alpha}{2}} \end{pmatrix}, \qquad |\phi_\pm\rangle = \frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{4}(1\pm1)}}{\sqrt{2}} \begin{pmatrix} 1 \\ \mp1 \end{pmatrix}, \tag{16}$$

respectively. Taking now as initial and final states the orthogonal states $|\phi_i\rangle$ and $|\phi_f\rangle$ as defined in (5), we compute

$$\mathrm{e}^{-\mathrm{i}th}|\phi_i\rangle = \mathrm{e}^{-\mathrm{i}r\cos\theta t} \begin{pmatrix} \cos\frac{\omega t}{2} \\ \mathrm{i}\sin\frac{\omega t}{2} \end{pmatrix}, \tag{17}$$

and recover from (1), with $\beta = 1$, the passage time $\tau = \pi/\omega$. This is what we expect from the general expression (7) and in fact it is the same expression as obtained in [6], where essentially the second equation in (1) was evaluated. On the other hand, if we now let the particle pass through the region in which the corresponding Hamiltonian becomes non-Hermitian, we may compute τ by analyzing (9) or possibly (10). Acting with $u(t,0) = \mathrm{e}^{-\mathrm{i}th}$ on the transformed initial state yields

$$\mathrm{e}^{-\mathrm{i}th}\eta|\phi_i\rangle = \frac{\mathrm{e}^{-\mathrm{i}tr\cos\theta}}{\sqrt{\cos\alpha}} \begin{pmatrix} \sin\frac{1}{2}(\alpha - t\omega) \\ \mathrm{i}\cos\frac{1}{2}(\alpha - t\omega) \end{pmatrix}. \tag{18}$$

When not acting on eigenstates with the operator $\mathrm{e}^{-\mathrm{i}th}$ or $\mathrm{e}^{-\mathrm{i}tH}$ one has to turn the infinite sum of operators into a matrix multiplication, see e.g. [6]. For this one can exploit the fact that any 2 × 2-matrix M can be decomposed in terms of Pauli matrices as $M = \mu_0\mathbb{I} + \mu\cdot\sigma$ with $\mu_i \in \mathbb{C}, i = 0, 1, 2, 3$. Having expressed h or H in this manner, the operation with $\mathrm{e}^{-\mathrm{i}th}$ or $\mathrm{e}^{-\mathrm{i}tH}$ on a state reduces to a simple matrix multiplication by using the identity $\mathrm{e}^{\varphi\mu\cdot\sigma} = \cos\varphi\mathbb{I} + \mathrm{i}\sin\varphi\mu\cdot\sigma$.

Using (18), the matrix element in (9) is computed to

$$\langle\phi_f|U(\tau,0)\phi_i\rangle_\eta = \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\tau r\cos\theta}\frac{\sin\left(\alpha - \frac{\tau\omega}{2}\right)}{\cos\alpha}. \tag{19}$$

Choosing the constant $\beta = 1$ and substituting (19) into (9), we compute with $\|\phi_f\|_\eta\|\phi_i\|_\eta = 1/\cos\alpha$ the passage time $\tau = \pi/\omega + 2\alpha/\omega$. This expression involves now the parameter α, which may be tuned to make τ arbitrarily small while keeping the transition frequency constant, as was first pointed out in [6].

There are two equivalent ways of looking this result. On one hand we may think that one has solved the quantum brachistochrone problem entirely within the framework of the Hermitian system for some states, which have no obvious intrinsic meaning without referring to the non-Hermitian counterpart. On the other hand we may think that one has solved the

Table 1. Passage times for various different initial and final states evolved by means of a PT-symmetric non-Hermitian time evolution operator. All matrix elements are computed with the η-inner product and states are normalized with the η-norm.

$|\psi_i\rangle$	$|\psi_f\rangle$	$U(\tau, 0)$	τ
$|\Phi_i\rangle$	$|\Phi_f\rangle$	$e^{-iH\tau}$	π/ω
$|\phi_i\rangle$	$|\phi_f\rangle$	$e^{-iH\tau}$	$\pi/\omega + 2\alpha/\omega$
$|\phi_i\rangle$	$|\Phi_f\rangle$	$e^{-iH\tau}$	$2\pi/\omega + \alpha/\omega$
$|\Phi_i\rangle$	$|\phi_f\rangle$	$e^{-iH\tau}$	$2\pi/\omega + \alpha/\omega$

time-dependent problem as outlined in the previous subsection involving the evolution with a non-Hermitian Hamiltonian between two orthonomal states in the Hermitian system.

Similarly, we may consider a situation in which the final state is constructed from eigenstates of the non-Hermitian system and compute instead

$$\langle \Phi_f | U(\tau, 0)\phi_i \rangle_\eta = i\, e^{-i\tau r \cos\theta} \frac{\cos\frac{1}{2}(\alpha - \tau\omega)}{\sqrt{\cos\alpha}}. \tag{20}$$

This yields for the same choice of $\beta = 1$ the passage time $\tau = 2\pi/\omega + \alpha/\omega$ with $\|\Phi_f\|_\eta \|\phi_i\|_\eta = 1/\sqrt{\cos\alpha}$. We summarize our findings for the different types of scenarios in table 1.

Next we demonstrate that such type of behaviour is not limited to PT-symmetric Hamiltonians, but can also be found for dissipative systems, i.e. genuinely non-Hermitian Hamiltonians with complex eigenvalues with negative imaginary part.

3. Non-Hermitian dissipative Hamiltonian systems

As argued above it is sufficient to consider a 2×2 matrix Hamiltonian. We will now consider two different types of dissipative systems, i.e. those which have real and those with complex transition frequencies.

3.1. Real transition frequency

Let us modify the Hamiltonian H in (14) slightly, such that it becomes a genuinely dissipative system. In order to achieve this we need to break the PT-symmetry not only for the wavefunction, but also for the Hamiltonian. Such type of Hamiltonians result for instance as effective Hamiltonians by coupling two non-degenerate states to some open channel as for instance explained in [23, 24]. We consider here such type of Hamiltonian of the particular form

$$\tilde{H} = \begin{pmatrix} E + \varepsilon & 0 \\ 0 & E - \varepsilon \end{pmatrix} - i\lambda \begin{pmatrix} r\, e^{i\theta} & s \\ s & r\, e^{-i\theta} \end{pmatrix}, \tag{21}$$

with $E, \varepsilon, r, s, \theta, \lambda \in \mathbb{R}$. Similarly as for (14) we will not provide here a concrete physical meaning for the parameters, as we would like to keep our treatment as generic as possible. Note that this Hamiltonian does not simply correspond to going to the regime of broken PT-symmetry for the Hamiltonian (14) of the previous section. Instead, in the simultaneous limit $E, \varepsilon \to 0$ and $\lambda \to i$, the dissipative Hamiltonian system $\tilde{H}$ reduces to the PT-symmetric Hamiltonian H.

The eigenvalues of $\tilde{H}$ are computed to

$$\tilde{\varepsilon}_\pm = E \pm \frac{\tilde{\omega}}{2} - ir\lambda \cos\theta, \tag{22}$$

with $\tilde{\omega} = \tilde{\varepsilon}_+ - \tilde{\varepsilon}_- = 2\sqrt{(\varepsilon + r\lambda\sin\theta)^2 - \lambda^2 s^2}$ denoting the transition frequency. For λ being restricted to the interval $-\varepsilon/(s + \sin\theta) \leqslant \lambda \leqslant \varepsilon/(s - \sin\theta)$, we can guarantee that $\tilde{\omega} \in \mathbb{R}$. In this parameter range the complex energy eigenvalues are indeed of the desired form of a decaying state, that is $\tilde{\varepsilon}_\pm = E_\pm - \mathrm{i}\Gamma/2$ with decay width $\Gamma = 2r\lambda\cos\theta \in \mathbb{R}^+$ when $-\pi/2 \leqslant \theta \leqslant \pi/2$. It is then useful to introduce the parameterization

$$\sin\tilde{\alpha} = \frac{s\lambda}{\varepsilon + r\lambda\sin\theta}, \tag{23}$$

such that $\tan\tilde{\alpha} = 2s\lambda/\tilde{\omega}$. The right eigenvectors of $\tilde{H}$ corresponding to the eigenvalues $\tilde{\varepsilon}_\pm$ in (22) may then be expressed as

$$|\tilde{\Phi}_\pm\rangle = \tfrac{1}{\sqrt{2}}(|\Phi_-\rangle_{\tilde{\alpha}} \pm \mathrm{i}|\Phi_+\rangle_{\tilde{\alpha}}), \tag{24}$$

where $|\Phi_\pm\rangle_\alpha$ is defined in equation (16). From the considerations in the previous section it is clear that the operator η is vital for the computations of the matrix elements occurring in the quantum brachistochrone problem, especially when one wishes to evolve eigenstates of a Hermitian Hamiltonian with a time-evolution operator associated with a non-Hermitian system. However, since for the case at hand the Hamiltonian $\tilde{H}$ is now genuinely complex there cannot exist any similarity transformation, which relates it to a Hermitian Hamiltonian. Nonetheless, we can use the other property of η, namely that it can be utilized to introduce a physically well-defined inner product. This means we can seek a transformation such that the eigenstates (24) become orthonormal with regard to this product. From (24) it is clear that we can take the same form for η, but only have to replace α by $\tilde{\alpha}$ to define a new $\tilde{\eta}$. With the help of this new operator we construct the eigenstates $|\tilde{\phi}_\pm\rangle = \tilde{\eta}|\tilde{\Phi}_\pm\rangle$, which yield indeed the desired orthogonality relations

$$\langle\tilde{\Phi}_n|\tilde{\Phi}_m\rangle_{\tilde{\eta}} = \langle\tilde{\eta}^{-1}\tilde{\phi}_n|\tilde{\eta}^{-1}\tilde{\phi}_m\rangle_{\tilde{\eta}} = \langle\tilde{\phi}_n|\tilde{\phi}_m\rangle = \delta_{nm}, \tag{25}$$

for $n, m \in \{+, -\}$. The states $\tilde{\phi}$ are eigenstates to the analogue of the Hermitian counterpart of a pseudo-Hermitian Hamiltonian. In fact, the adjoint action of $\tilde{\eta}$ diagonalizes $\tilde{H}$ as

$$\tilde{h} = \tilde{\eta}\tilde{H}\tilde{\eta}^{-1} = \begin{pmatrix} E - \mathrm{i}\frac{\Gamma}{2} - \frac{\tilde{\omega}}{2} & 0 \\ 0 & E - \mathrm{i}\frac{\Gamma}{2} + \frac{\tilde{\omega}}{2} \end{pmatrix}. \tag{26}$$

Obviously we have now $\tilde{h} \neq \tilde{h}^\dagger$, but $\tilde{h}$ has the same eigenvalues as $\tilde{H}$, because it lies in the same similarity class.

3.1.1. $\tilde{\Phi} \to \tilde{\Phi}$ via $\tilde{U}$. We are now in the position to solve the quantum brachistochrone problem for dissipative non-Hermitian Hamiltonians. For this we note first that when we try to compute the passage time τ directly from the relation (1) with $\beta = 1$, it will turn out to be complex. This would of course always be the case when the transition amplitude for all real values of t is smaller than β. Therefore in order to find a physical solution we need to reformulate the quantum brachistochrone problem slightly to accommodate also the dissipative non-Hermitian Hamiltonians. A natural expression to consider is one which features explicitly the decay width Γ, such as

$$|\langle\tilde{\phi}_f|\tilde{u}(\tau, 0)\tilde{\phi}_i\rangle| = |\langle\tilde{\Phi}_f|\tilde{U}(\tau, 0)\tilde{\Phi}_i\rangle_{\tilde{\eta}}| = \tilde{\beta}\,\mathrm{e}^{-\frac{\Gamma\pi}{2\tilde{\omega}}}, \tag{27}$$

for normalized initial and final states with $0 \leqslant \tilde{\beta} \leqslant 1$. This means for stable particles, i.e. $\Gamma \to 0$, we recover the expression (4). In analogy to the normalized eigenstates of the Hermitian Hamiltonian h in (5), we take now the initial and final states to be

$$|\tilde{\phi}_{f/i}\rangle = \frac{\tilde{\eta}}{\sqrt{2}}(|\tilde{\Phi}_-\rangle \pm \mathrm{i}|\tilde{\Phi}_+\rangle) = \frac{1}{\sqrt{2}}(|\tilde{\phi}_-\rangle \pm \mathrm{i}|\tilde{\phi}_+\rangle). \tag{28}$$

Table 2. Passage times for various different initial and final states evolved by means of a dissipative non-Hermitian time evolution operator with real transition frequency. All matrix elements are computed with the $\tilde{\eta}$-inner product and states are normalized with the η-norm.

$|\psi_i\rangle$	$|\psi_f\rangle$	$\tilde{U}(\tau,0)$	τ
$|\tilde{\Phi}_i\rangle$	$|\tilde{\Phi}_f\rangle$	$e^{-it\tilde{H}}$	$\pi/\tilde{\omega}$
$|\tilde{\phi}_i\rangle$	$|\tilde{\phi}_f\rangle$	$e^{-it\tilde{H}}$	$\pi/\tilde{\omega}$
$|\tilde{\phi}_i\rangle$	$|\tilde{\Phi}_f\rangle$	$e^{-it\tilde{H}}$	$\pi/\tilde{\omega}$
$|\tilde{\Phi}_i\rangle$	$|\tilde{\phi}_f\rangle$	$e^{-it\tilde{H}}$	$\pi/\tilde{\omega}$

Assuming at first no time dependence in the Hamiltonian, we compute

$$\langle\tilde{\phi}_f|e^{-it\tilde{h}}\tilde{\phi}_i\rangle = \tfrac{1}{2}\,e^{-i\tilde{\varepsilon}_- t}(1 - e^{-i\tilde{\omega}t}). \tag{29}$$

Solving then (27) for the passage time in complete analogy to (4) with $\tilde{\beta} = 1$ yields the same expression for the passage time, namely $\tau = \pi/\tilde{\omega}$. Alternatively we could have also computed the second expression in (27) using $\tilde{U}(t,0) = e^{-it\tilde{H}}$, which would of course lead to the same result. Possibly more interesting passage times can be obtained when we evolve the states (28) with the analogue to the non-Hermitian time evolution.

3.1.2. $\tilde{\Phi} \to \tilde{\Phi}$ via $\tilde{u}$. We may assume now an explicit time dependence as in section 2.3 and try to evolve the states $|\tilde{\phi}_i\rangle$ by means of a time-evolution operator involving the Hamiltonians $\tilde{H}$ in (21). For that situation to make sense we need to consider the $\tilde{\eta}$-inner products and generalize (9) to

$$\frac{|\langle\tilde{\phi}_f|\tilde{U}(\tau,0)\tilde{\phi}_i\rangle_{\tilde{\eta}}|}{\|\tilde{\phi}_f\|_{\tilde{\eta}}\|\tilde{\phi}_i\|_{\tilde{\eta}}} = \frac{|\langle\tilde{\eta}\tilde{\phi}_f|\tilde{u}(\tau,0)\tilde{\eta}\tilde{\phi}_i\rangle|}{|\tilde{\eta}\tilde{\phi}_f||\tilde{\eta}\tilde{\phi}_i|} = \tilde{\beta}\,e^{-\frac{\Gamma\pi}{2\tilde{\omega}}}, \tag{30}$$

by introducing the decay width on the right-hand side, similarly as we extended (4) to (27). This way the passage time will result to be real. Assuming a simple stepfunction time-dependence in (8), we compute

$$e^{-it\tilde{h}}\tilde{\eta}|\tilde{\phi}_i\rangle = \frac{e^{-it(E-i\lambda r\cos\theta)}\sqrt{\cos\tilde{\alpha}}}{\sqrt{2}\left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\right)}\begin{pmatrix} -e^{it\tilde{\omega}/2} \\ i\,e^{-it\tilde{\omega}/2} \end{pmatrix} \tag{31}$$

and the matrix element

$$\langle\tilde{\phi}_f|e^{-it\tilde{H}}\tilde{\phi}_i\rangle_{\tilde{\eta}} = \langle\tilde{\eta}\tilde{\phi}_f|e^{-it\tilde{h}}\tilde{\eta}\tilde{\phi}_i\rangle \tag{32}$$

$$= -i\,e^{-it(E-i\lambda r\cos\theta)}\sin\frac{t\tilde{\omega}}{2}. \tag{33}$$

Since $\|\tilde{\phi}_f\|_{\tilde{\eta}}\|\tilde{\phi}_i\|_{\tilde{\eta}} = 1$ in (30), we obtain with $\tilde{\beta} = 1$ the passage time $\tilde{\tau} = \pi/\tilde{\omega}$. Likewise we may evaluate analogues to the other matrix elements computed in section 2.3. Our results are summarized in table 2.

Thus somewhat surprisingly, despite the fact the matrix elements are somwhat different, the normalization factors compensate for this and in all cases the passage time results to $\tilde{\tau} = \pi/\tilde{\omega}$. This might suggest that we should really attribute the possibility of tunable passage times to the $\mathcal{PT}$-symmetry. However, we have not yet studied the possibility when $\tilde{\omega} \notin \mathbb{R}$.

3.2. Complex transition frequency

Often it is not even possible to restrict the parameter range so nicely like in the previous subsection as to ensure that $\tilde{\omega} \in \mathbb{R}$. Instead considering the scenario of leaving this regime

for the above example, let us consider a different system, which does not even possess such a regime and can be found for instance in [25]

$$\hat{H} = \begin{pmatrix} E + \varepsilon & 0 \\ 0 & E - \varepsilon \end{pmatrix} - \frac{\mathrm{i}\lambda}{2} \begin{pmatrix} 2\cos^2\phi & \sin 2\phi \\ \sin 2\phi & 2\sin^2\phi \end{pmatrix}, \tag{34}$$

with $E, \varepsilon \in \mathbb{R}$ and $\lambda, \phi \in \mathbb{C}$. In [25] the special case $\lambda, \phi \in \mathbb{R}$ was treated for a concrete physical setting. Here we keep these parameters completely generic. The eigenvalues of $\hat{H}$ are

$$\hat{\varepsilon}_{\pm} = E \pm \frac{\hat{\omega}}{2} - \mathrm{i}\frac{\lambda}{2}, \tag{35}$$

with energy gap $\hat{\omega} = \hat{\varepsilon}_+ - \hat{\varepsilon}_- = \sqrt{4\varepsilon^2 - \lambda^2 - 4\mathrm{i}\varepsilon\lambda\cos 2\phi}$. Now we have lost the property of the transition frequency to be real. We parameterize instead

$$\mathrm{e}^{-\mathrm{i}\hat{\alpha}} = \frac{-\mathrm{i}\lambda\sin 2\phi}{2\varepsilon + \hat{\omega} - \mathrm{i}\lambda\cos 2\phi}, \tag{36}$$

such that the eigenvectors can simply be taken to be $|\hat{\Phi}_{\pm}\rangle = |\Phi_{\pm}\rangle_{\hat{\alpha}}$. Replacing now also in η, as defined in (15), the parameter α by $\hat{\alpha}$ defines a new operator $\hat{\eta}$. We employ this operator to construct the Hamiltonian

$$\hat{h} = \hat{\eta}\hat{H}\hat{\eta}^{-1} = \begin{pmatrix} E - \mathrm{i}\frac{\lambda}{2} & -\frac{\hat{\omega}}{2} \\ -\frac{\hat{\omega}}{2} & E - \mathrm{i}\frac{\lambda}{2} \end{pmatrix}, \tag{37}$$

with eigenvectors $|\hat{\phi}_{\pm}\rangle := |\phi_{\pm}\rangle$ as defined in (16). We may now compute the passage time in a very similar fashion as for the $\mathcal{PT}$-symmetric example, keeping however in mind that the transition frequency $\hat{\omega}$ as well as the parameter $\hat{\alpha}$ are complex. A consequence of the latter is that $\hat{\eta}$ is no longer Hermitian, that is $\hat{\eta}^{\dagger} \neq \hat{\eta}$. For convenience we introduce the abbreviations $\hat{\omega} = \hat{\omega}_r + \mathrm{i}\hat{\omega}_i$, $\hat{\alpha} = \hat{\alpha}_r + \mathrm{i}\hat{\alpha}_i$ and $\lambda = \lambda_r + \mathrm{i}\lambda_i$ for $\lambda_{i/r}, \hat{\omega}_{i/r}, \hat{\alpha}_{i/r} \in \mathbb{R}$.

3.2.1. $\hat{\Phi} \to \hat{\Phi}$ via $\hat{U}$. It is straightforward to compute the square of the transition probability

$$|\langle\hat{\phi}_f|\mathrm{e}^{-\mathrm{i}t\hat{h}}\hat{\phi}_i\rangle|^2 = \tfrac{1}{2}\,\mathrm{e}^{-\lambda_r t}[\cosh(t\hat{\omega}_i) - \cos(t\hat{\omega}_r)], \tag{38}$$

when taking the initial and final states to be the orthonormal states as defined in (5), with $|\phi_{\pm}\rangle \to |\hat{\phi}_{\pm}\rangle$. There are now various possibilities we can equate this to and subsequently compute a passage time τ. However, choosing a generic $0 \leqslant \beta \leqslant 1$ as for instance in equation (1) would lead to very involved solutions, due to the transcendental nature of this equation. Furthermore, since we are now dealing with a dissipative system the right-hand side of (38) only reaches a maximum $\beta' < 1$. Thus the choice $\beta > \beta'$ would lead to unphysical complex passage times. In order to keep matters simple, we could make a natural choice and just take this maximum of the right-hand side of (38), i.e. $\beta = \beta'$. However, due to the transcendental nature of this equation, there is no elegant analytic solution for τ to this equation. Instead we try to make this solution to be as closely related to previously computed expressions as possible and choose the right-hand side such that the passage time becomes $\tau = \pi/\hat{\omega}_r$ leading to a particular $\breve{\beta} \leqslant \beta'$. We note that this choice does not lead to a loss of generality with regard to the main aim of our investigation, which is to seek passage times which can be made arbitrarily small. Taking any other value below the maximum will simply lead to another definite value for τ, when keeping $\hat{\omega}$ constant.
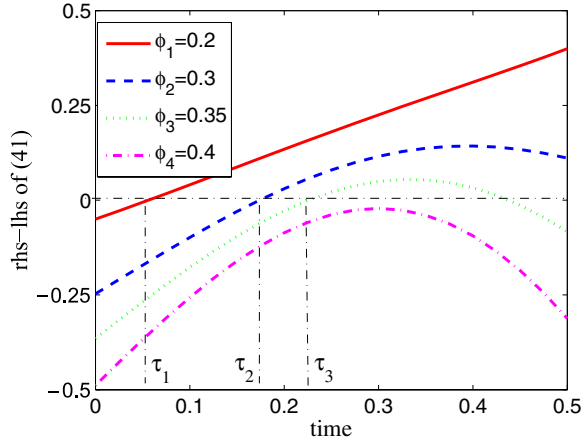
9

Figure 1. Right-hand side (rhs) minus the left-hand side (lhs) of equation (41) as a function of time t with fixed values of $\varepsilon = 2.5$, $E = 2$, $\omega = 4.87 - i3.31$ for $\phi_1 = 0.2$, $\lambda_1 = 3.5$; $\phi_2 = 0.3$, $\lambda_2 = 3.73 + i0.2$; $\phi_3 = 0.35$, $\lambda_3 = 3.87 + i0.34$; $\phi_4 = 0.4$, $\lambda_4 = 4.02 + i0.52$. The corresponding passage times τ are simply taken to be the smallest values of t for which the rhs–lhs = 0.
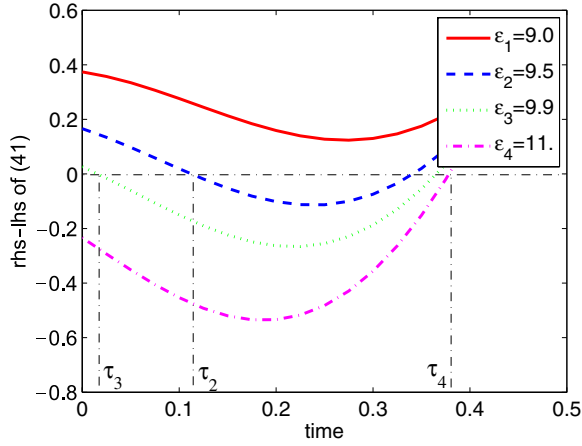

Figure 2. Right-hand side minus the left-hand side of equation (41) as a function of time t with fixed values of $\phi = 0.2$, $E = 2$, $\omega = 6.99 - i0.46$ for $\varepsilon_1 = 9.0$, $\lambda_1 = 1.85 - i14.84$; $\varepsilon_2 = 9.5$, $\lambda_2 = 2.72 - i16.32$; $\varepsilon_3 = 9.9$, $\lambda_3 = 3.41 - i17.29$; $\varepsilon_4 = 11.0$, $\lambda_4 = 5.01 - i19.62$.

3.2.2. $\hat{\Phi} \rightarrow \hat{\Phi}$ via $\hat{u}$. As in all previous scenarios we take the special case $\hat{\Phi} \rightarrow \hat{\Phi}$ via $\hat{U}$ as a benchmark for the choice of the constant $\beta = \check{\beta}$, since it can be dealt with analytically. Let us now compute

$$|\langle \hat{\phi}_f | e^{-it\hat{H}} \hat{\phi}_i \rangle_{\hat{\eta}}|^2 = \frac{\cosh(t\hat{\omega}_i) - \cos(2\hat{\alpha}_r - t\hat{\omega}_r)}{2e^{\lambda_r t} \cos\hat{\alpha} \cos\hat{\alpha}^*}. \tag{39}$$

With

$$\|\hat{\phi}_f\|_{\hat{\eta}}^2 \|\hat{\phi}_i\|_{\hat{\eta}}^2 = \frac{\cosh^2 \hat{\alpha}_i}{\cos\hat{\alpha} \cos\hat{\alpha}^*} \tag{40}$$

and the choice of $\beta = \check{\beta}$ as discussed in the previous section, namely taking it as the right-hand side of (38) with $t \rightarrow \pi/\hat{\omega}_r$, the quantum brachistochrone problem amounts to solving

10

$$\frac{\cosh(t\hat{\omega}_i) - \cos(2\hat{\alpha}_r - t\hat{\omega}_r)}{\cosh^2 \hat{\alpha}_i (1 + \cosh(\pi\hat{\omega}_i/\hat{\omega}_r))} = e^{\lambda_r(t - \pi/\hat{\omega}_r)} \tag{41}$$

in this case. Since this is a transcendental equation, we can not solve it in complete generality and we are therefore content to discuss some numerical solutions. For this purpose we can solve the equation of the transition frequency for the coupling constant $\lambda = -2i\varepsilon \cos\phi + \sqrt{4\varepsilon^2 \sin^2 2\phi - \omega^2}$ and express it as a function of ϕ, ε and ω. Since we want to keep ω constant, we investigate (41) as a function of time t by varying ϕ and λ or ε and λ see figure 1 or figure 2, respectively.

The analysis of equation (41), as depicted in figures 1 and 2 demonstrates that it is possible to find tunable passage times for Hamiltonian systems of the type (34). The precise dependence of the system on the parameters is rather involved in this case, but our analysis demonstrates that it is possible to approach $\tau \approx 0$. Similar conclusions can be drawn when changing $|\phi_f\rangle$ to $|\Phi_f\rangle$ or $|\phi_i\rangle$ to $|\Phi_i\rangle$, respectively.

4. Conclusion

In [6] the authors have extended the formulation of the quantum brachistochrone problem by allowing that the time-evolution operator may be associated to non-Hermitian Hamiltonians which are $\mathcal{PT}$-symmetric, that is those with real eigenvalues. Here we did not insist on the $\mathcal{PT}$-symmetry of the Hamiltonian, but allowed this symmetry to be completely broken. In order to take the most extreme case, we did not just spontaneously break the $\mathcal{PT}$-symmetry for the wavefunction, which would result in complex conjugate pairs for the energy eigenfunction, but we allowed in addition that the $\mathcal{PT}$-symmetry is also broken for the Hamiltonian. Thus we have considered an effective Hamiltonian whose energy eigenvalues have a negative imaginary part, such that it is associated to dissipative systems. We found the same intriguing feature as observed in [6] for the quantum brachistochrone problem for $\mathcal{PT}$-symmetric non-Hermitian Hamiltonians, namely that the passage time can be made arbitrarily small also for non-Hermitian Hamiltonians associated to these types of systems. Our observations suggest that this type of phenomenon may occur when one projects between orthonormal states, which are not eigenstates of the non-Hermitian Hamiltonian associated to the time-evolution operator, irrespective of whether this Hamiltonian is $\mathcal{PT}$-symmetric or not.

Clearly there are various open questions to be answered. First of all, it would be highly desirable to have a more formal and generic proof for these phenomena, rather than case-by-case studies. This holds for the $\mathcal{PT}$-symmetric case treated in [6] as well as for the case presented here. In addition, one could make the above considerations more involved by allowing more complicated time-dependences rather than the simple stepfunction and study other possibilities in (8). For such more realistic scenarios we may have to resort to a perturbative treatment using (13).

Acknowledgments

References

[1] Goldstein H F and Bender C M 1986 Relativistic brachistochrone *J. Math. Phys.* **27** 507–11

[2] Miyake A and Wadati M 2001 Geometric strategy for the optimal quantum search *Phys. Rev.* A **64** 042317

[3] Brody D C 2003 Elementary derivation for passage times *J. Phys. A: Math. Gen.* **36** 5587–93

[4] Carlini A, Hosoya A, Koike T and Okudaira Y 2006 Time-optimal quantum evolution *Phys. Rev. Lett.* **96** 060503

[5] Brody D C and Hook D W 2006 On optimum Hamiltonians for state transformations *J. Phys. A: Math. Gen.* **39** L 167–70

[6] Bender C M, Brody D C, Jones H F and Meister B K 2007 Faster than Hermitian quantum mechanics *Phys. Rev. Lett.* **98** 040403

[7] Figueira de Morisson Faria C and Fring A 2007 Non-Hermitian Hamiltonians with real eigenvalues coupled to electric fields: from the time-independent to the time dependent quantum mechanical formulation *Laser Phys.* **17** 424–37 (*Preprint* quant-ph/0609096)

[8] Geyer H, Heiss D and Znojil M (guest editors) 2006 Special issue dedicated to the physics of non-hermitian operators (PHHQP IV) (University of Stellenbosch, South Africa, 23–25 Nov 2005) *J. Phys. A: Math. Gen.* **39** 9965–10261

[9] Bender C M 2007 Making sense of non-Hermitian Hamiltonians *Rep. Prog. Phys.* **70** 947–1018 (*Preprint* hep-th/0703096)

[10] Znojil M (guest editors) 2006 Special issue: pseudo-Hermitian Hamiltonians in quantum physics *Czech. J. Phys.* **56** 885–1064

[11] Scholtz F G, Geyer H B and Hahne F 1992 Quasi-Hermitian operators in quantum mechanics and the variational principle *Ann. Phys.* **213** 74–101

[12] Mostafazadeh A 2002 Pseudo-Hermiticity versus PT symmetry: the necessary condition for the reality of the spectrum *J. Math. Phys.* **43** 205–14

[13] Mostafazadeh A 2002 Pseudo-Hermiticity versus PT-symmetry II: a complete characterization of non-hermitian Hamiltonians with a real spectrum *J. Math. Phys.* **43** 2814–6

[14] Mostafazadeh A 2002 Pseudo-Hermiticity versus PT-symmetry III: equivalence of pseudo-hermiticity and the presence of anti-linear symmetries *J. Math. Phys.* **43** 3944–51

[15] Mostafazadeh A 2003 Exact PT-symmetry is equivalent to Hermiticity *J. Phys. A: Math. Gen.* **36** 7081–92

[16] Bender C M, Brody D C and Jones H F 2002 Complex extension of quantum mechanics *Phys. Rev. Lett.* **89** 270401(4)

[17] Persson E, Gorin T and Rotter I 1996 Decay rates of resonance states at high level density *Phys. Rev.* E **54** 3339–51

[18] Rotter I and Sadreev A F 2005 Avoided level crossing, diabolic points in the complex plane in a double quantum dot *Phys. Rev.* E **71** 0362271

[19] Figueira de Morisson Faria C and Fring A 2006 Time evolution of non-hermitian Hamiltonian systems *J. Phys. A: Math. Gen.* **39** 9269–89

[20] Fring A, Kostrykin V and Schrader R 1996 On the absence of bound-state stabilization through short ultra-intense fields *J. Phys. B: At. Mol. Opt. Phys.* **29** 5651–567

[21] Figueira de Morisson Faria C, Fring A and Schrader R 1999 Analytical treatment of stabilization *Laser Phys.* **9** 379–87

[22] Figueira de Morisson Faria C, Fring A and Schrader R 2000 Existence criteria for stabilization from the scaling behaviour of ionization probabilities *J. Phys. B: At. Mol. Opt. Phys.* **33** 1675–168

[23] Friedrich H and Wintgen D 1985 Interfering resonances and bound states in the continuum *Phys. Rev.* A **32** 3231–43

[24] Friedrich H 2006 *Theoretical Atomic Physics* (Berlin: Springer)

[25] Heiss W D, Müller M and Rotter I 1998 Collectivity, phase transitions and exceptional points in open quantum systems *Phys. Rev.* E **58** 2894–901